An integral equation formulation of the Navier–Stokes equations

James S. Uhlman *

Engineering Technology Center, Anteon Corporation, Mystic, CT 06355, USA

Abstract

A set of coupled integral equations is derived from the incompressible Navier–Stokes equations and the continuity equation. These equations are based on a velocity–vorticity–total pressure formulation and are exact. The equations consist of a generalization of the Biot–Savart law for the determination of the velocity, an integral expression of the momentum equation for the determination of the vorticity and a boundary integral equation for the determination of the total pressure. The equations possess a number of interesting properties, including the absence of spatial derivatives and the fact that the total pressure is only required on the boundary of the fluid domain. In addition, since for steady flows the vorticity is present in all volume integrals, the domain of integration in this case is restricted to the region of nonzero vorticity. All boundary conditions, and in particular the far-field boundary condition, are naturally incorporated in the formulation.

Keywords: Navier–Stokes equations; Integral equations

1. Introduction

In the course of various investigations it became apparent that it might be possible to express the equations of motion of an incompressible fluid solely in terms of integral equations. In fact, it turns out to be possible to derive such a set of coupled integral equations in what may be called the velocity–vorticity–total pressure formulation. This article presents the resultant equations and discusses their properties with a view toward their numerical solution. Details of the derivation may be found in Uhlman [2].

2. Mathematical formulation

We shall consider the flow of an incompressible fluid in an unbounded domain in body-fixed coordinates (formulations for bounded flows may also be readily derived). The velocity may then be expressed as the sum

$$\mathbf{U}_\infty + \mathbf{u} \tag{1}$$

where in an unbounded fluid domain $\mathbf{u}$ is the disturbance velocity and $\mathbf{U}_\infty$ is the freestream velocity. The governing differential equations are then, the continuity equation

$$\nabla \cdot \mathbf{u} = 0 \tag{2}$$

and the incompressible Navier–Stokes equations

$$\rho \frac{\partial (\mathbf{U}_\infty + \mathbf{u})}{\partial t} + \rho (\mathbf{U}_\infty + \mathbf{u}) \cdot \nabla \mathbf{u} = -\nabla p + \mu \nabla^2 \mathbf{u} \tag{3}$$

The Navier–Stokes equations may also be written in the form

$$\rho \frac{\partial (\mathbf{U}_\infty + \mathbf{u})}{\partial t} + \nabla H - \rho (\mathbf{U}_\infty + \mathbf{u}) \times \boldsymbol{\omega} = -\mu \nabla \times \boldsymbol{\omega} \tag{4}$$

where the total pressure, H, is defined as

$$H = (p - p_\infty) + \tfrac{1}{2}\rho \left[(\mathbf{U}_\infty + \mathbf{u}) \cdot (\mathbf{U}_\infty + \mathbf{u}) - \mathbf{U}_\infty \cdot \mathbf{U}_\infty\right] \tag{5}$$

and

$$\boldsymbol{\omega} = \nabla \times \mathbf{u} \tag{6}$$

is the vorticity.

The derivation of the integral equation formulation requires two integral identities. The first identity is the vector identity

$$\beta \mathbf{a} = -\oiint_S \left\{(\mathbf{n} \cdot \mathbf{a})\,\mathbf{G} - \mathbf{G} \times (\mathbf{n} \times \mathbf{a})\right\} \mathrm{d}S + \iiint_V \left\{(\nabla \cdot \mathbf{a})\,\mathbf{G} - \mathbf{G} \times (\nabla \times \mathbf{a})\right\} \mathrm{d}V \tag{7}$$

* Correspondence address: Box B9, 650 Ten Rod Road, North Kingston, RI 02852, USA. Tel.: +1 (401) 267-0182; Fax: +1 (401) 294-8720; E-mail: juhlman@anteon.com

Computational Fluid and Solid Mechanics
K.J. Bathe (Editor)

where if the fluid domain is denoted by V and its boundary is S then

$$\beta = \begin{cases} 4\pi & \text{in } V \\ 2\pi & \text{on } S \\ 0 & \text{in } V^c \end{cases} \tag{8}$$

and $\mathbf{G}$ is a vector Green's function which may be taken to be

$$\mathbf{G} = \frac{\mathbf{R}}{R^3} \tag{9}$$

The second integral identity is a generalization of Green's third identity [1]

$$\beta\phi = \oiint_S \left\{ \frac{\partial \phi}{\partial n} G - \phi \frac{\partial G}{\partial n} \right\} \mathrm{d}S - \iiint_V \nabla^2 \phi G \, \mathrm{d}V \tag{10}$$

where G is a scalar Green's function which may be taken to be

$$G = \frac{1}{R} \tag{11}$$

These identities holds for any scalar and vector fields which are differentiable and for which the integrals exist.

These integral identities may be applied to the Navier–Stokes equations [2] to yield the set of coupled integral equations in the velocity–vorticity–total pressure formulation as,

$$\beta \mathbf{u} = -\oiint_S \left\{ \frac{(\mathbf{n}\cdot\mathbf{u})\,\mathbf{R}}{R^3} - \frac{\mathbf{R}\times(\mathbf{n}\times\mathbf{u})}{R^3} \right\} \mathrm{d}S - \iiint_V \frac{(\mathbf{R}\times\boldsymbol{\omega})}{R^3} \mathrm{d}V \tag{12}$$

and

$$\begin{aligned} \beta\boldsymbol{\omega} = & -\oiint_S \left\{ \frac{(\mathbf{n}\cdot\boldsymbol{\omega})\,\mathbf{R}}{R^3} - \frac{\mathbf{R}\times(\mathbf{n}\times\boldsymbol{\omega})}{R^3} \right\} \mathrm{d}S \\ & + \frac{1}{\mu} \oiint_S H \frac{\mathbf{R}\times\mathbf{n}}{R^3} \mathrm{d}S \\ & + \frac{1}{\nu} \oiint_S \left[\mathbf{n} \times \frac{\partial(\mathbf{U}_\infty + \mathbf{u})}{\partial t} \right] \left(\frac{1}{R} \right) \mathrm{d}S \\ & - \frac{1}{\nu} \iiint_V \left\{ \frac{\partial \boldsymbol{\omega}}{\partial t} \left(\frac{1}{R} \right) + \frac{\mathbf{R}\times\left[(\mathbf{U}_\infty + \mathbf{u})\times\boldsymbol{\omega}\right]}{R^3} \right\} \mathrm{d}V \end{aligned} \tag{13}$$

and

$$\begin{aligned} \beta H + \oiint_S H \frac{\partial}{\partial n}\left(\frac{1}{R}\right) \mathrm{d}S = & \\ & -\rho \oiint_S \left[\mathbf{n} \cdot \frac{\partial(\mathbf{U}_\infty + \mathbf{u})}{\partial t} \left(\frac{1}{R} \right) + \nu \frac{\mathbf{R}\cdot(\mathbf{n}\times\boldsymbol{\omega})}{R^3} \right] \mathrm{d}S \\ & + \rho \iiint_V \frac{\mathbf{R}\cdot\left[(\mathbf{U}_\infty + \mathbf{u})\times\boldsymbol{\omega}\right]}{R^3} \mathrm{d}V \end{aligned} \tag{14}$$

It should be noted that a two-dimensional version of these equations has also been formulated.

3. Discussion

Equations (12) through (14) represent an integral equation reformulation of the Navier–Stokes equations. They contain no spatial derivatives and only require knowledge of the total pressure on the boundary of the fluid domain. Since all the volume integrals present in the equations contain the vorticity and since the far-field boundary condition is incorporated in the formulation, it is immediately apparent that only the rotational regions of the flow need be considered.

These equations also have interesting properties when examined with an eye toward their numerical solution. They form a coupled set of second kind Fredholm integral equations which are well-known for their generally good numerical properties. Since the volume integrals are only non-zero over the rotational portion of the flow and the far-field boundary condition is built in, the solution domain may be restricted to the rotational region, thereby shrinking the computational domain.

The matrix equations formed by discretization of these equations will be full. This situation would make their solution computationally infeasible for large problems were it not for the fact that their kernel functions are of a form which will allow the application of accelerated methods such as the Fast Multipole Method (FMM) of Greengard [3] and Rokhlin [4]. With the introduction of acceleration, the fullness of the matrix becomes an asset in that it allows all regions of the grid to communicate with one another directly at each iteration. This feature should reduce the number of iterations required for solution, relative to the number necessary in the solution of sparse matrices. It should be noted that the multigrid methods introduced for the acceleration of the solution of sparse matrices are closely related the acceleration methods required to achieve computational competitiveness in the present approach.

Work is presently ongoing to develop a numerical solution method for the Navier–Stokes equations in the present integral equation form. Results of that effort are anticipated soon.

References

[1] Carrier GF, Pearson CE. Partial Differential Equations. New York: Academic Press, 1976.

[2] Uhlman JS. An integral equation formulation of the equations of motion of an incompressible fluid, NUWC-NPT Technical Report 10,086. Naval Undersea Warfare Center, Newport, RI, 1992.

[3] Greengard LF. The Rapid Evaluation of Potential Fields in Particle Systems. Cambridge, MA: MIT Press, 1988.

[4] Rokhlin V. Rapid solution of integral equations of classical potential theory. J Comput Phys 1985;60:187–207.